Hi all,

Here is the math for solving the complex river game drawn in Figure 7.15. The results and the players' distributions at each point in the game tree are discussed in detail in Chapter 7, and understanding them gives us a lot of insight into the structure of solid river play. However, the actual math to solve them didn't have the work-to-usefulness ratio needed to be included in the book proper.

To the people viewing this before reading the book: this is not particularly representative of its contents. In the book, we focus very heavily on building intuition and understanding in order to play exploitatively, not doing algebra. That said, I hope this will be useful to some particularly ambitious students of the game. See also the supplemental files which will allow you to play around with these equations and their solutions using a computer algebra system.

Best wishes, Will

Appendix A

Solving the big river game

Consider the decision tree in Figure 7.15. The structures of the GTO strategies for this game are shown in Figure 7.16. We discuss these strategies in Section 7.3.3. The twelve threshold hands which describe the equilibrium strategies are defined in the same section. Here we present the indifference equations which can be used to solve the game. The general solutions to these equations depend on the equity distributions and are a bit too ugly to print even in the case of symmetric distributions. They can be readily obtained from the equations using a computer algebra package.

To write down the indifference equations, we assume the following.

$$0 < h^b_{bbf} < \mathrm{EQ}_{\mathrm{SB}}(h^s_{bbf}) < h^b_{cf} < h^b_{bcr} < \mathrm{EQ}_{\mathrm{SB}}(h^s_{sd}) < h^b_{cc} < \mathrm{EQ}_{\mathrm{SB}}(h^s_{vbf}) < h^b_{bc} < 1$$

$$0 < \mathrm{EQ}_{\mathrm{SB}}(h^s_{bbf}) < \mathrm{EQ}_{\mathrm{SB}}(h^s_{f}) < \mathrm{EQ}_{\mathrm{SB}}(h^s_{br}) < h^b_{cc} < h^b_{vbf} < \mathrm{EQ}_{\mathrm{SB}}(h^s_{c}) < h^b_{bc} < 1$$

These assumptions are built in to the dotted line drawing in Figure 7.16. They are all satisfied by the solutions to the symmetric distribution case. If they are not satisfied for some particular equity distributions of interest, the frequencies and equities which make up the indifference equations below might need to be re-derived.

First of all, we can find some frequencies by reading them right off Figure 7.16 that will make the indifference equations easier to follow. We will indicate frequencies by using `typewriter text`. The notation describing the cut-off hands is explained on page 265. First, we have the overall fraction of hands with which the BB bets, checks, and check-calls at his first decision point.

$$\begin{aligned}
\texttt{bb_chk} &= 1 - h^b_{bc} + h^b_{cc} - h^b_{bbf} \\
\texttt{bb_bet} &= 1 - \texttt{bb_chk} \\
\texttt{bb_cc} &= h^b_{cc} - h^b_{bcr}
\end{aligned}$$

Then, we have some frequencies assuming the BB already checked: the SB's total frequencies for betting and checking as well as those for bet-folding and bet-calling.

$$\begin{aligned}
\texttt{sb_chk} &= h^s_{sd} - h^s_{bbf} \\
\texttt{sb_bet} &= 1 - h^s_{sd} + h^s_{bbf} \\
\texttt{sb_bf} &= h^s_{vbf} - h^s_{sd} + h^s_{bbf} \\
\texttt{sb_bc} &= 1 - h^s_{vbf}
\end{aligned}$$

Now, the BB's check-folding, check-calling, and check-raising frequencies, assuming he has already checked, are as follows.

$$\texttt{bb\_cf\_after\_chk} = \frac{h^b_{cf} - h^b_{bbf}}{\texttt{bb\_chk}}$$
$$\texttt{bb\_cc\_after\_chk} = \frac{h^b_{cc} - h^b_{bcr}}{\texttt{bb\_chk}}$$
$$\texttt{bb\_cr\_after\_chk} = \frac{1 - h^b_{bc} + h^b_{bcr} - h^b_{cf}}{\texttt{bb\_chk}}$$

Now, assume the BB has bet. Then, we have the frequencies with which the SB folds to the bet, calls the bet, and raises the bet.

$$\texttt{sb\_ftb} = h^s_f$$
$$\texttt{sb\_cb} = h^s_c - h^s_{br}$$
$$\texttt{sb\_rb} = 1 - h^s_c + h^s_{br} - h^s_f$$

Finally, we have the BB's fold-to-raise and call-raise frequencies, assuming the SB has raised.

$$\texttt{bb\_ftr} = \frac{h^b_{vbf} - h^b_{cc} + h^b_{bbf}}{-h^b_{cc} + h^b_{bbf} + h^b_{bc}}$$
$$\texttt{bb\_callr} = \frac{h^b_{bc} - h^b_{vbf}}{-h^b_{cc} + h^b_{bbf} + h^b_{bc}}$$

Now we can write down the indifference at each threshold hand.

1. At h^b_{bbf}, $\mathrm{EV_{BB}}$(bet-fold) = $\mathrm{EV_{BB}}$(check-fold):

$$\texttt{sb\_ftb}(S + P) + \texttt{sb\_cb}(S - B) + \texttt{sb\_rb}(S - B) = S$$

since the BB always loses with h^b_{bbf} if he plays check-fold since the SB only checks to showdown with better hands.

2. At h^b_{cf}, $\mathrm{EV_{BB}}$(check-fold) = $\mathrm{EV_{BB}}$(check-raise):

$$\texttt{sb\_bet}(S) = \texttt{sb\_bf}(S + P + B) + \texttt{sb\_bc}(S - C)$$

where we have cancelled on both sides the contribution arising from the times that the SB checks back since this is the same regardless of whether the BB had planned to fold or raise had he faced a bet.

3. At h^b_{bcr}, $\mathrm{EV_{BB}}$(check-raise) = $\mathrm{EV_{BB}}$(check-call):

$$\texttt{sb\_bc}(S - C) + \texttt{sb\_bf}(S + P + B) = \texttt{sb\_bet}\left(S - B + (P + 2B)\left(\frac{h^s_{bbf}}{h^s_{bbf} + 1 - h^s_{sd}}\right)\right)$$

where we have again cancelled the contributions arising from the times the SB checks behind since that is the same on both sides.

4. At h^b_{cc}, $\text{EV}_{\text{BB}}(\text{bet-fold}) = \text{EV}_{\text{BB}}(\text{check-call})$:

$$\begin{aligned}&\texttt{sb\_ftb}(S+P) + \texttt{sb\_cb}\left(S - B + (P+2B)\left(\frac{\text{EQ}_{\text{BB}}(h^b_{cc}) - h^s_{br}}{h^s_c - h^s_{br}}\right)\right) + \texttt{sb\_rb}(S-B)\\ &= \texttt{sb\_bet}\left(S - B + (P+2B)\left(\frac{h^s_{bbf} + \text{EQ}_{\text{BB}}(h^b_{cc}) - h^s_{sd}}{h^s_{bbf} + 1 - h^s_{sd}}\right)\right) + \texttt{sb\_chk}(S+P)\end{aligned}$$

The fraction on the left-hand side here is the BB's equity with h^b_{cc} versus the SB's calling range. Similarly, the fraction on the right-hand side is the BB's equity with h^b_{cc} when he checks and faces a bet.

5. At h^b_{vbf}, $\text{EV}_{\text{BB}}(\text{bet-fold}) = \text{EV}_{\text{BB}}(\text{bet-call})$:

$$\texttt{sb\_rb}(S-B) = \texttt{sb\_rb}\left(S - C + (P+2C)\left(\frac{h^s_{br} - h^s_f}{h^s_{br} - h^s_f + 1 - h^s_c}\right)\right)$$

Here, we have cancelled the terms arising from the times the SB calls or folds to a bet since these are the same on both sides.

6. At h^b_{bc}, $\text{EV}_{\text{BB}}(\text{bet-call}) = \text{EV}_{\text{BB}}(\text{check-raise})$:

$$\begin{aligned}&\texttt{sb\_ftb}(S+P) + \texttt{sb\_cb}(S+P+B) + \texttt{sb\_rb}\left(S - C + (P+2C)\left(\frac{h^s_{br} - h^s_f + \text{EQ}_{\text{BB}}(h^b_{bc}) - h^s_c}{h^s_{br} - h^s_f + 1 - h^s_c}\right)\right)\\ &= \texttt{sb\_chk}(S+P) + \texttt{sb\_bf}(S+P+B) + \texttt{sb\_bc}\left(S - C + (P+2C)\left(\frac{\text{EQ}_{\text{BB}}(h^b_{bc}) - h^s_{vbf}}{1 - h^s_{vbf}}\right)\right)\end{aligned}$$

The first fraction here is the BB's equity with h^b_{bc} versus the SB's raising range, and the second is his equity verus the SB's bet-calling range.

7. At h^s_{bbf}, $\text{EV}_{\text{SB}}(\text{bet-fold}) = \text{EV}_{\text{SB}}(\text{check})$:

$$\begin{aligned}&\texttt{bb\_cf\_after\_chk}(S+P) + \texttt{bb\_cr\_after\_chk}(S-B) + \texttt{bb\_cc\_after\_chk}(S-B)\\ &= S + P\left(\frac{\text{EQ}_{\text{SB}}(h^s_{bbf}) - h^b_{bbf}}{\texttt{bb\_chk}}\right)\end{aligned}$$

since the SB never wins after betting with h^s_{bbf} if the BB calls. The fraction at the end is the SB's equity with h^s_{bbf} versus the BB's river checking range.

8. At h^s_{sd}, $\text{EV}_{\text{SB}}(\text{bet-fold}) = \text{EV}_{\text{SB}}(\text{check})$:

$$\begin{aligned}&\texttt{bb\_cf\_after\_chk}(S+P) + \texttt{bb\_cc\_after\_chk}\left(S - B + (P+2B)\left(\frac{\text{EQ}_{\text{SB}}(h^s_{sd}) - h^b_{bcr}}{\texttt{bb\_cc}}\right)\right)\\ &+ \texttt{bb\_cr\_after\_chk}(S-B)\\ &= S + P\left(\frac{\text{EQ}_{\text{SB}}(h^s_{sd}) - h^b_{bbf}}{1 - h^b_{bc} + h^b_{cc} - h^b_{bbf}}\right)\end{aligned}$$

The two fractions here are the SB's equities with h^s_{sd} versus the BB's river check-calling range and the BB's river checking range, respectively.

9. At h^s_{vbf}, $\text{EV}_{\text{SB}}(\text{bet-fold}) = \text{EV}_{\text{SB}}(bet - call)$:

$$S - B = S - C + (P + 2C)\left(\frac{h^b_{bcr} - h^b_{cf}}{h^b_{bcr} - h^b_{cf} + 1 - h^b_{bc}}\right)$$

   Again, the contributions to the SB's EV arising from the times the BB check-folds or check-calls on the river are the same on both sides. Only the SB's river check-raising tendencies matter when comparing the SB's bet-folding and bet-calling options.

10. At h^s_f, the SB is facing a bet, and $\text{EV}_{\text{SB}}(\text{fold}) = \text{EV}_{\text{SB}}(\text{raise})$:

$$S = \texttt{bb\_ftr}(S + P + B) + \texttt{bb\_callr}(S - C)$$

11. At h^s_{br}, the SB is facing a bet, and $\text{EV}_{\text{SB}}(\text{raise}) = \text{EV}_{\text{SB}}(\text{call})$:

$$\texttt{bb\_ftr}(S + P + B) + \texttt{bb\_callr}(S - C) = S - B + (P + 2B)\left(\frac{h^b_{bbf}}{h^b_{bbf} + h^b_{bc} - h^b_{cc}}\right)$$

   where the fraction is the SB's equity with h^s_{br} facing the BB's river leading range.

12. At h^s_c, the SB is again facing a bet, and $\text{EV}_{\text{SB}}(\text{raise}) = \text{EV}_{\text{SB}}(\text{call})$:

$$\texttt{bb\_ftr}(S + P + B) + \texttt{bb\_callr}\left(S - C + (P + 2C)\left(\frac{\text{EQ}_{\text{SB}}(h^s_c) - h^b_{vbf}}{h^b_{bc} - h^b_{vbf}}\right)\right)$$
$$= S - B + (P + 2B)\left(\frac{h^b_{bbf} + \text{EQ}_{\text{SB}}(h^s_c) - h^b_{cc}}{h^b_{bbf} + h^b_{bc} - h^b_{cc}}\right)$$

   The first fraction here is the SB's equity versus the range with which the BB calls a river raise, and the second is his equity versus all of BB's river leading range.

Solutions for this in the symmetric distributions case for a sampling of bet and raise sizings are shown in Figure 7.17.